Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER
International Journal of Solids and Structures 42 (2005) 5920–5930
INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES
www.elsevier.com/locate/ijsolstr

Stabilization of beam parametric vibrations with shear deformations and rotary inertia effects

Andrzej Tylikowski *

Institute of Machine Design Fundamentals, Warsaw University of Technology, Narbutta 84, 02-524 Warszawa, Poland

Received 27 November 2004
Available online 5 May 2005

Abstract

The purpose of this theoretical work is to present a stabilization problem of beam with shear deformations and rotary inertia effects. A velocity feedback and particular polarization profiles of piezoelectric sensors and actuators are introduced. The structure is described by partial differential equations with time-dependent coefficient including transverse and rotary inertia terms, general deformation state with interlaminar shear strains. The first order deformation theory is utilized to investigate beam vibrations. The beam motion is described by the transverse displacement and the slope. The almost sure stochastic stability criteria of the beam equilibrium are derived using the Liapunov direct method. If the axial force is described by the stationary and continuous with probability one process the classic differentiation rule can be applied to calculate the time-derivative of functional. The particular problem of beam stabilization due to the Gaussian and harmonic forces is analyzed in details. The influence of the shear deformations, rotary inertia effects and the gain factors on dynamic stability regions is shown.

Keywords: Vibration control; Piezoelectric actuators; Shear deformations; Rotary inertia; Liapunov method

1. Introduction

In the paper, theoretical fundamentals of stabilization of beam with shear deformations and rotary inertia effects are presented. The piezoelectric layers are glued to the both sides of the beam compressed by time-dependent axial forces. A velocity feedback and particular polarization profiles of piezoelectric sensors and actuators are introduced. The structure is described by partial differential equations including

* Tel.: +48 22 660 8244; fax: +48 22 660 8622.
E-mail address: aty@simr.pw.edu.pl

0020-7683/$ - see front matter
doi:10.1016/j.ijsolstr.2005.03.023

transverse and rotary inertia terms, general deformation state with interlaminar shear strains. A viscous model of external damping with the constant proportionality coefficient is assumed to describe a dissipation of the structure energy both in the transverse and rotary motion. The first order deformation theory is utilized to investigate beam vibrations. The beam motion is described by the transverse displacement. In order to estimate deviations of solutions from the plane equilibrium state a scalar measure of distance equal to the square root of the suitable functional is introduced. The almost sure stochastic stability criteria of the structure equilibrium are derived using the Liapunov direct method. If the axial force is described by the stationary and continuous with probability one process the classic differentiation rule can be applied to calculate the time-derivative of functional. In order to find an exponential one-side estimation the calculus of variation is used. The associated Euler equations in the form of system of differential equations are solved analytically and the stabilization problem is reduced to transcendental algebraic inequality with respect the exponent of estimation. The particular problem of beam stabilization due to the Gaussian and harmonic forces is analyzed in details. The influence of the shear deformations, rotary inertia effects and the gain factors on dynamic instability regions is shown.

The problem of correction of shear on the transverse beam vibration goes back to the work of Timoshenko in 1921 (see Timoshenko and Gere, 1961). The influence of shear deformation on the natural frequencies of laminated rectangular plates was examined by Dave and Craig (1985). Shear deformation effects on thermal buckling of cross-ply composite laminates were analyzed by Mannini (1997). Timoshenko beam-bending solutions in terms of Euler–Bernoulli solutions were given by Wang (1995). The influence of transverse shear on dynamic stability domains was studied by Pavlović et al. (2001). The thermally induced parametric vibrations of laminated plates with shear effects due to the time-dependent temperature with Gaussian and harmonic distributions were analyzed (Tylikowski, 2003). Analytical solutions for the length and position of strain-induced patch actuators for the static adjustments of Timoshenko's beam deflection were presented by Ang et al. (2000). The effects of the feedback control gain on the parametric vibrations of a beam with piezoelectric layers compressed by harmonic axial force were examined by Chen et al. (2002).

2. Basic assumptions, definitions

Consider the beam of length l, width b, and thickness h_b, loaded by axial time-dependent force with piezoelectric layers mounted on each of two opposite sides. The beam is simply supported on both ends. The piezoelectric layers are assumed to be bonded on the beam surfaces and the mechanical properties of the bonding material are represented by the effective damping coefficient calculated from the rule of mixture. The damping coefficient is a linear function of both the beam and bonding layer damping coefficients. It is assumed that the transverse motion dominates the axial vibrations. The thickness of the actuator and the sensor is denoted by h_a and h_s, respectively. Assuming a negligible stiffness of the piezolayer in comparison with that of the beam and the changing width $b_s(X)$ of the sensor, the changing width $b_a(X)$ of the actuator the influence of the piezoelectric actuator on the beam can be reduced to bending moment M_x distributed along the actuator.

The transverse motion of the beam is described by the uniform equation with time-dependent coefficients. Its trivial solution $w = 0$ corresponds to the undisturbed state. The trivial solution is called stable in Liapunov sense if the following definition is satisfied:

$$\bigwedge_{\varepsilon>0} \bigvee_{\delta>0} \|w(0,.)\| < \delta \Rightarrow \bigwedge_{t>0} \|w(t,.)\| < \varepsilon \tag{1}$$

where $\|\cdot\|$ is a measure of distance of disturbed solution w from the equilibrium state. When the axial force is a stochastic processes we call the trivial solution almost sure asymptotically stable if

$$P\left\{\lim_{t\to\infty}\|w(t,.)\| = 0\right\} = 1 \quad (2)$$

3. Sensor and actuator equations

Sensor electric displacement in direction perpendicular to the beam surface is given by

$$D_3 = -e_{31}\varepsilon_1 \quad (3)$$

where e_{13} is the piezoelectric stress/charge coefficient, and ε_1 sensor strain. Expressing strains by the beam curvatures and the distance from the neutral axis we integrate the electric displacement over the sensor area

$$D_3 = -d_s\frac{(h_s+h_b)E_s}{2}\int_0^l b_s(X)w_{,XX}\,dX \quad (4)$$

where d_s is the piezoelectric strain/charge coefficient of sensor. Finally, the sensor voltage is calculated using the formula for a flat capacitor

$$V_s = -d_s\frac{(h_s+h_b)E_s h_s}{2\varepsilon_{33}A_s}\int_0^l b_s(X)w_{,XX}\,dX \quad (5)$$

where A_s is the effective sensor area and ε_{33} is the permittivity coefficient. Using the velocity feedback control the voltage applied to the actuator is

$$V_a = \frac{K_a}{C}\frac{dV_s}{dt} \quad (6)$$

The control bending moment can be expressed by the actuator stress σ_a, moment arm $h_b + h_a$, and the cross-section area $t_a b(x)$ of the actuator in the following way:

$$M^e = d_{a31}V_a h_a\frac{h_b+h_a}{2}b_a(X) \quad (7)$$

4. Closed loop dynamics equation of beam with rotational inertia effect

Consider the beam with rotational inertia effect loaded axially by a time-dependent force with piezoelectric layers mounted on each of the two opposite sides of the beam. The piezoelectric layers are assumed to be bonded on the beam surfaces and the mechanical properties of the bonding material are represented by the effective damping coefficient. The sensing and actuating effects of piezoelectric layers are used to stabilize both the free vibrations due to initial disturbances and parametric vibrations excited by the oscillating axial force. Assuming the negligible stiffness of the sensor in comparison with that of the beam the influence of the piezoelectric actuator on the beam is reduced to a bending moment M^e distributed along the beam

$$\rho A w_{,TT} + 2\rho A\beta^* w_{,T} + EJw_{,XXXX} - EJ\rho A w_{,XXTT} - 2\rho I\beta^* w_{,XXT} + (F_0 + F(T))w_{,XX} + M^e_{,XX} = 0$$
$$0 < X < l \quad (8)$$

where ρ—mass density, EJ—bending stiffness, A—cross-section area, β^*—damping coefficient, F_0—constant component of axial force, $F(T)$—time-dependent component, M^e—moment of electric origin, w—beam transverse displacement. Introducing dimensionless coordinates

$$T = tk_t = l^2\sqrt{\rho A/EJ}t \quad X = xl \quad (9)$$

and denotations

$$\beta = \beta^* k_t \quad \varepsilon = I/Al^2 \quad v = w_{,t} \tag{10}$$

we obtain the equation of beam motion with piezoelectric sensors and actuators

$$v_{,t} + 2\beta v - \varepsilon(v_{,txx} + 2\beta v_{,xx}) + w_{,xxxx} + (f_0 + f(t))w_{,xx} + m^{\mathrm{e}}_{,xx} = 0 \tag{11}$$

where the moment of piezoelectric origin is as follows:

$$m^{\mathrm{e}} = \gamma^* b_{\mathrm{a}}(x) \int_0^1 w_{,xxt} b_{\mathrm{s}}(x)\,\mathrm{d}x \tag{12}$$

Due to the simply supported edges we have the following boundary conditions:

$$w(0,t) = w(1,t) = 0 \quad w_{,xx}(0,t) = w_{,xx}(1,t) = 0 \tag{13}$$

5. Closed loop dynamics equation of beam with shear deformation effect

Consider the beam with shear deformation effect loaded axially by a time-dependent force with piezoelectric layers mounted on each of the two opposite sides of the beam. Introducing the same assumptions as in Section 4 the influence of the piezoelectric actuator on the beam is reduced to a bending moment M^{e} distributed along the beam.

Using the first order deformation theory based on Timoshenko approach we obtain the following equation of beam motion with neglecting the rotational inertia effect:

$$\rho A w_{,TT} + 2\rho A \beta^* w_{,T} + EJ w_{,XXXX} - EJ\rho A w_{,XXTT} - 2\beta^* \frac{EJ\rho A}{\kappa} w_{,XXT} + (F_0 + F(T))(w_{,XX} - EJ/\rho w_{,XXXX}) + M^{\mathrm{e}}_{,XX} = 0 \quad 0 < X < l \tag{14}$$

where ρ—mass density, EJ—bending stiffness, A—cross-section area, κ—shear stiffness, β*—damping coefficient, F_0—constant component of axial force, $F(T)$—time-dependent component of axial force, M^{e}—moment of electric origin, u—beam transverse displacement. Introducing dimensionless coordinates

$$T = k_t t = l^2\sqrt{\rho A/EJ}\,t \quad X = xl \tag{15}$$

and denotations

$$\beta = \beta^* k_t \quad \varepsilon = EJ/\kappa l^2 \quad v = u_{,t} \tag{16}$$

we obtain the equation of beam motion with piezoelectric sensors and actuators

$$v_{,t} + 2\beta v - \varepsilon(v_{,txx} + 2\beta v_{,xx}) + w_{,xxxx} + (f_0 + f(t))(w_{,xx} - \varepsilon w_{,xxxx}) + m^{\mathrm{e}}_{,xx} = 0 \tag{17}$$

Assuming simply supported edges we have the boundary conditions (13).

6. Stability analysis of closed loop system

6.1. Harmonic parametric excitation

Assume that the beam is subjected to the harmonic force

$$f(t) = a\cos pt \tag{18}$$

Consider the frequency p of parametric excitation in the range corresponding to the lowest resonance ω. Determine the polarization profiles of the sensor and actuator as follows:

$$\Phi_M^{\mathrm{s}} = \alpha_M \sin \pi x \quad \Phi_M^{\mathrm{a}} = \beta_M \sin \pi x \quad M = 1 \tag{19}$$

Therefore, the stability problem is reduced to the following Mathieu equation:

$$\frac{\mathrm{d}^2 u}{\mathrm{d}t^2} + 2\gamma \frac{\mathrm{d}u}{\mathrm{d}t} + \omega_0^2 (1 + \mu \cos pt) u = 0 \tag{20}$$

where the effective damping coefficients γ is defined as follows:

$$\gamma = \beta + \gamma^* \pi^4 / 4(1 + \varepsilon \pi^2) \tag{21}$$

The free vibration frequency ω_0 and the excitation coefficient μ have different forms for the beam with rotational inertia effect

$$\begin{aligned} \omega_0^2 &= \frac{\pi^2(\pi^2 - f_0)}{1 + \varepsilon \pi^2} \\ \mu &= \frac{a}{\pi^2 - f_0} \end{aligned} \tag{22}$$

and for the beam with shear deformation effect

$$\begin{aligned} \omega_0^2 &= \pi^2 \left(\frac{\pi^2}{1 + \varepsilon \pi^2} - f_0 \right) \\ \mu &= \frac{a}{\frac{\pi^2}{1 + \varepsilon \pi^2} - f_0} \end{aligned} \tag{23}$$

The meaning of parameter ε is also different (cf. (10) and (16)). The main instability region is shown in Fig. 1.

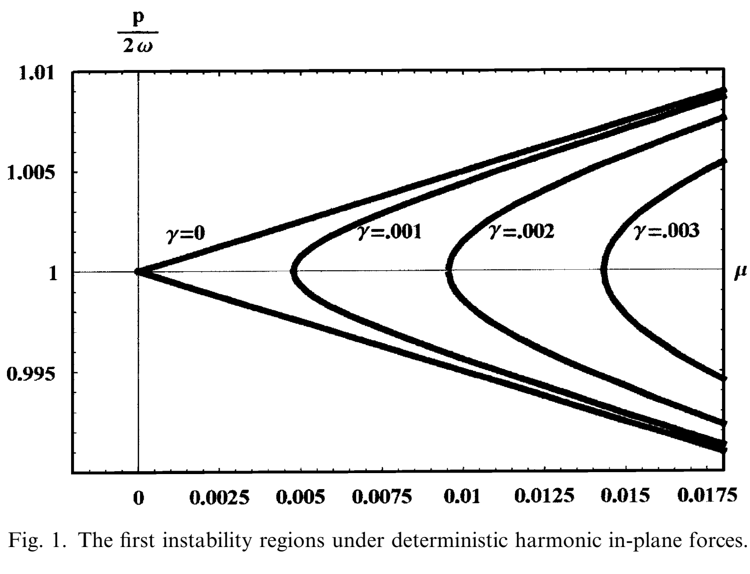

Fig. 1. The first instability regions under deterministic harmonic in-plane forces.

6.2. Stochastic parametric excitation of beam with rotational inertia effect

Consider the axial excitation in the form of physically realizable stochastic process with known probability distribution. In order to analyze stochastic instability we introduce Liapunov functional of the form

$$V = \frac{1}{2}\int_0^1 [v^2 + 2\beta v w + 2\beta^2 w^2 + w_{,xx}^2 - f_0 w_{,x}^2 + \varepsilon(v_{,x}^2 + 2\beta v_{,x} w_{,x} + 2\beta^2 w_{,x}^2)]\,\mathrm{d}x \tag{24}$$

If the classical condition for static buckling is fulfilled, the functional (22) satisfies positive-definiteness condition, and the measure of distance can be chosen as the square root of the functional

$$\|w(.,t)\| = \sqrt{V} \tag{25}$$

If the trajectories of the forces are physically realizable ergodic processes the classical calculus is applied to calculation of the time-derivative of Eq. (24). Upon differentiation with respect to time, substituting dynamic equation (11) and using beam boundary conditions we obtain the time-derivative of functional in the form

$$\frac{\mathrm{d}V}{\mathrm{d}t} = -2\lambda V + 2U \tag{26}$$

where the auxiliary functional U is defined

$$U = \int_0^1 \left[\beta^2 v w + \beta^3 w^2 + \varepsilon(\beta^2 v_{,x} u_{,x} + \beta^3 u_{,x}^2) - \frac{f(t)}{2}(\beta w + v) w_{,xx} - \frac{1}{2}(\beta w + v) M_{,xx}^{\mathrm{e}}\right] \mathrm{d}x \tag{27}$$

We look for a function χ defined as a minimum over all admissible functions w and v of the ratio U/V

$$\lambda = \min_{w,v} \frac{U}{V} \tag{28}$$

In order to derive the associated Euler equations we have to determine the polarization profiles of sensor and actuators. We assume that widths correspond to the modal shape function of the chosen eigenfrequency ω_M

$$\begin{aligned} b_M^{\mathrm{s}}(x) &= \sin M\pi x \\ b_M^{\mathrm{a}}(x) &= \sin M\pi x \end{aligned} \tag{29}$$

The choice of modal shape functions in the form (29) corresponds to the one-mode control. We calculate $\delta(U - \lambda V) = 0$ and obtain the Euler equations in the form ($M = 1$)

$$\beta(\beta - \lambda)(w - \varepsilon w_{,xx}) - \lambda(v - \varepsilon v_{,xx}) - \frac{f(t)}{2} w_{,xx} + \frac{\gamma^*}{2}\pi^4 \sin \pi x \int_0^1 (2v + \beta w) \sin \pi x \,\mathrm{d}x = 0 \tag{30}$$

$$\beta(\beta - \lambda)[(v - \varepsilon v_{,xx}) + 2\beta(w - \varepsilon w_{,xx})] - f(t)\left(\beta w_{,xx} + \frac{1}{2} v_{,xx}\right) - \lambda(w_{,xxxx} - f_0 w_{,xx}) + \frac{\gamma^*}{2}\pi^4 \sin \pi x \times \int_0^1 v \sin \pi x \,\mathrm{d}x = 0 \tag{31}$$

We look for solutions of equations (30) and (31) in the form satisfying conditions (13)

$$w(x) = S_n \sin n\pi x \tag{32}$$

$$v(x) = T_n \sin n\pi x \tag{33}$$

Substituting solutions (32) and (33) into system (30) and (31) we obtain the linear homogenous system of equations with respect to S_n, T_n. Selecting the determinant of coefficients equal to zero gives the function λ_n. Due to the orthogonality property of sine functions the function λ_n for $n = 1$ and $n > 1$ are calculated from different formulae

$$\lambda_1 = \frac{a\gamma + \sqrt{a^2\gamma^2 + a(1+\varepsilon\pi^2)[\beta^2(1+\varepsilon\pi^2) + \pi^2 f(t)/2 - \beta\gamma]^2}}{a(1+\varepsilon\pi^2)} \tag{34}$$

where $a = (1+\varepsilon\pi^2)\beta^2 + \pi^2(\pi^2 - f_0)$ and $\gamma = \gamma^*\pi^4/4$

$$\lambda_n = \frac{|\beta^2(1+\varepsilon n^2\pi^2) + f(t)n^2\pi^2/2|}{\sqrt{(1+\varepsilon n^2\pi^2)[\beta^2(1+\varepsilon n^2\pi^2) + n^2\pi^2(n^2\pi^2 - f_0)]}} \quad n > 1 \tag{35}$$

The feedback gain factor of modal control is denoted by γ. The function λ is defined in the following way:

$$\lambda = \min_{n=1,2,3,\ldots} \{\lambda_n\} \tag{36}$$

Using the property of function λ in Eq. (26) leads to the first order differential inequality

$$\frac{dV}{dt} \leqslant -2(\beta - \lambda(t))V \tag{37}$$

Solving inequality (37) and assuming the ergodicity of time-dependent component $f(t)$ of axial force we obtain the lower estimation of functional $V(t)$ as follows:

$$V(t) \geqslant V(0)\exp[(E(\lambda) - \beta)t] \tag{38}$$

Thus, the equilibrium state of a beam with rotational inertia and with velocity feedback (the trivial solution of Eq (11)) is almost sure asymptotically unstable if the damping coefficient β is smaller than the mathematical expectation of function λ.

$$\beta \leqslant E(\lambda) \tag{39}$$

The instability regions with one-mode control as functions of constant component of the axial force, loading variance σ, damping coefficient β and gain factor γ are calculated numerically and shown in Fig. 2.

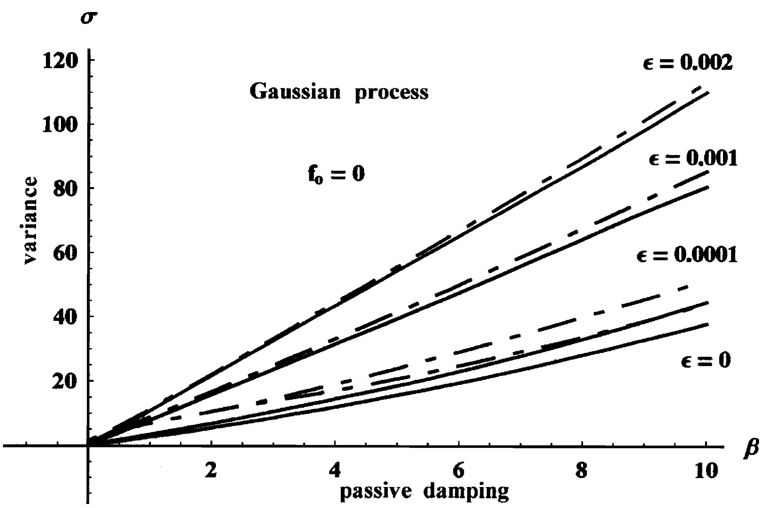

Fig. 2. Effect of passive damping coefficient β on the boundaries of instability domain of beam with rotational inertia with different $\varepsilon = I/(Al^2)$: $\gamma = 0$—continuous line, $\gamma = 0.05$—broken line.

In order to improve the feedback stabilization effect the two-mode control can be applied. The element λ_2 of sequence $\{\lambda_n\}$ in Eq. (36) has to be modified in the following way:

$$\lambda_2 = \frac{a\gamma + \sqrt{a^2\gamma^2 + a(1+4\varepsilon\pi^2)[\beta^2(1+4\varepsilon\pi^2) + 4\pi^2 f(t)/2 - \beta\gamma]^2}}{a(1+4\varepsilon\pi^2)} \tag{40}$$

where $a = (1+4\varepsilon\pi^2)\beta^2 + 4\pi^2(4\pi^2 - f_0)$ and $\gamma = 4\gamma^*\pi^4$. The first element λ_1 is the same (Eq. (34)) and Eq. (35) is applied for $n > 2$. From comparison shown in Fig. 3 we see that the double-mode control significantly decreases instability regions. This influence is more pronounced for small parameter ε, that corresponds to smaller rotational inertia.

The instability domains do not change significantly when going from the Gaussian process to the harmonic one for the wide range of parameters (cf. Fig. 4).

The effect of constant tensile force on instability regions is stabilizing especially with the control turn on (Fig. 5).

6.3. Stochastic parametric excitation of beam with shear deformation effect

Consider the axial excitation in the form of physically realizable stochastic process with known probability distribution. In order to analyze stochastic instability of trivial solution of Eq. (17) we introduce Liapunov functional of the form

$$V = \frac{1}{2}\int_0^1 \left[v^2 + 2\beta v w + 2\beta^2 w^2 + w_{,xx}^2 - f_0(w_{,x}^2 + \varepsilon w_{,xx}^2) + \varepsilon(v_{,x}^2 + 2\beta v_{,x} w_{,x} + 2\beta^2 w_{,x}^2)\right] \mathrm{d}x \tag{41}$$

We remember that the dimensionless parameter ε, in contradistinction to Section 6.2, is defined as follows: $\varepsilon = EJ/\kappa l^2$. The functional (37) is positively definite if the Timoshenko's condition is fulfilled

$$f_0 < \pi^2/(1+\varepsilon\pi^2) \tag{42}$$

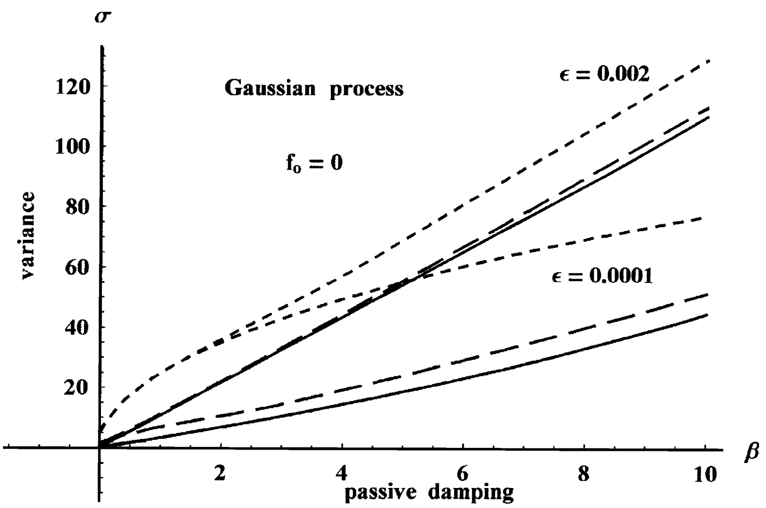

Fig. 3. Comparison of instability domains of beam with rotational inertia with different $\varepsilon = I/(Al^2)$: no control—continuous line, one-mode control—broken line, double-mode control—dotted line.

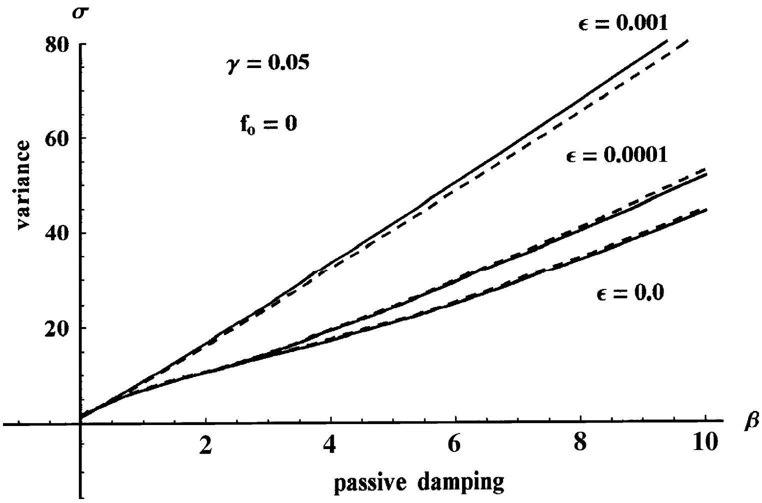

Fig. 4. Comparison of instability domains of beam with rotational inertia with different $\varepsilon = I/(Al^2)$: Gaussian process—continuous line, harmonic process—broken line.

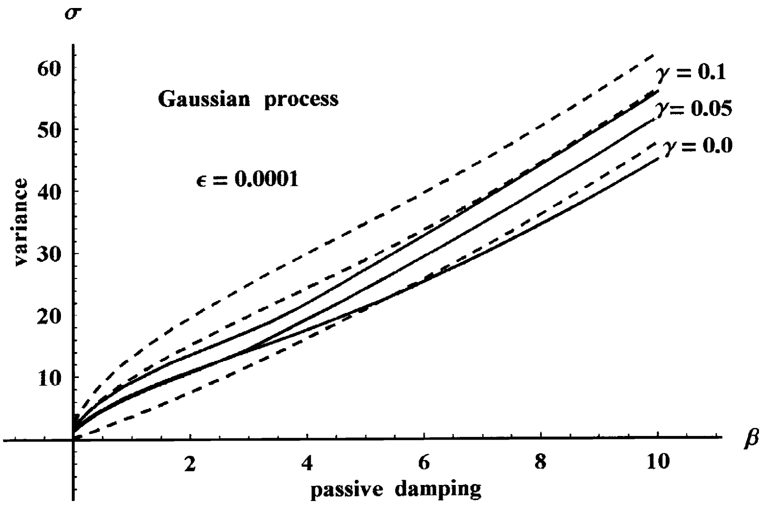

Fig. 5. Effect of static load component f_0 on the instability domains with different feedback control gain: $f_0 = 0$—continuous line, $f_0 = -10$—broken line.

and a distance between the disturbed solution and the trivial one can be defined as the square root of the functional (cf. Eq. (25)). As the classical differentiation principle can be applied, we obtain the differential of the functional in the form

$$\frac{\mathrm{d}V}{\mathrm{d}t} = -2\beta V + 2U \tag{43}$$

where U is given by

$$U = \int_0^1 \left[\beta^2 vw + \beta^3 w^2 + \varepsilon(\beta^2 v_{,x} w_{,x} + \beta^3 w_{,x}^2) - \frac{f(t)}{2}(\beta w + v)(w_{,xx} - \varepsilon w_{,xxxx}) - \frac{1}{2}(\beta w + v)M^{\mathrm{e}}_{,xx}\right] \mathrm{d}x \tag{44}$$

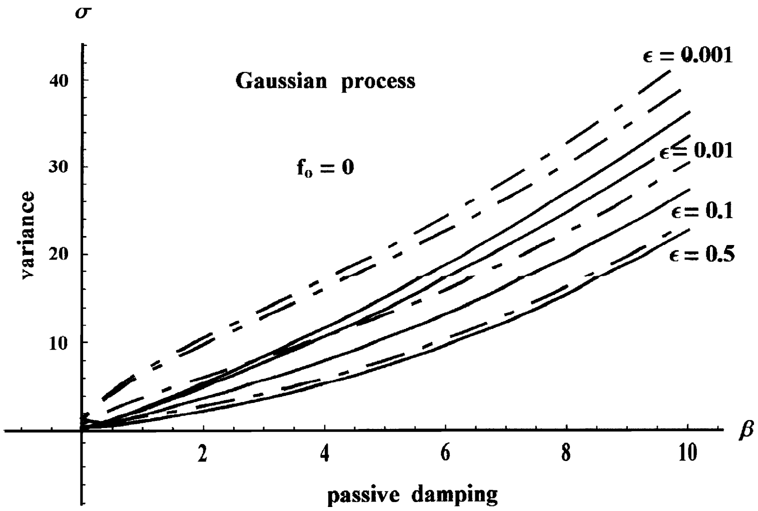

Fig. 6. Effect of passive damping coefficient β on the boundaries of instability domain of beam with shear deformation with different $\varepsilon = EJ/(\kappa l^2)$: $\gamma = 0$—continuous line, $\gamma = 0.05$—broken line.

Applying the standard stability analysis (cf. Section 6.2) we have the lower estimation of functional U

$$U < \lambda(t)V \tag{45}$$

Using the variational calculus we find the function $\lambda(t)$. Finally, the almost sure instability of the trivial solution $w = 0$ has the form (39) with function λ defined as follows:

$$\lambda = \min_{n=1,2,3,\ldots} \{\lambda_n\}$$

$$\lambda_1 = \frac{a\gamma + \sqrt{a^2\gamma^2 + a(1+\varepsilon\pi^2)[\beta^2(1+\varepsilon\pi^2) + \pi^2 f(t)/2 - \beta\gamma]^2}}{a(1+\varepsilon\pi^2)} \tag{46}$$

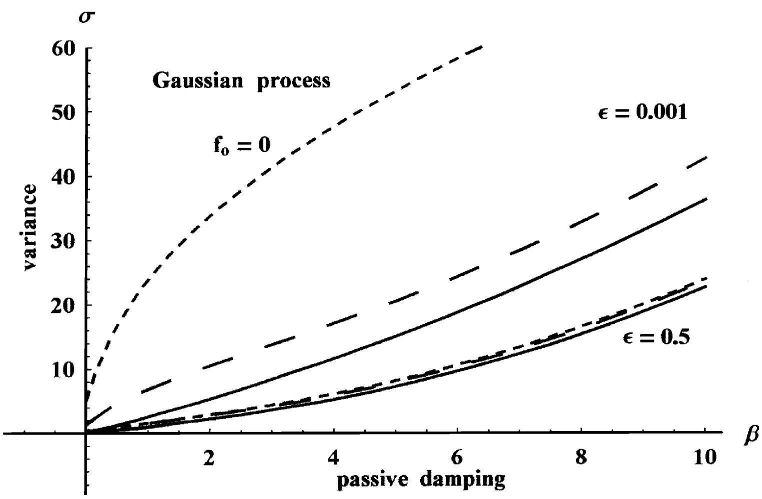

Fig. 7. Comparison of instability domains of beam with shear deformation with different $\varepsilon = EI/(\kappa l^2)$: no control—continuous line, one-mode control—broken line, double-mode control—dotted line.

where $a = \beta^2 + \pi^2(\pi^2/(1 + \varepsilon\pi^2) - f_0)$ and $\gamma = \gamma^*\pi^4/(4(1 + \varepsilon\pi^2))$

$$\lambda_n = \frac{|\beta^2 + f(t)n^2\pi^2/2|}{\sqrt{\beta^2 + n^2\pi^2[n^2\pi^2/(1 + \varepsilon n^2\pi^2) - f_0]}} \quad n > 1 \tag{47}$$

The stability regions as functions of the axial force variance σ, damping coefficient β and gain factor γ are calculated numerically and shown in Fig. 6. The instability regions are situated over lines. The increase of shear stiffness stabilizes significantly parametric vibrations. The double-mode control dramatically decreases instability domains (Fig. 7).

7. Conclusions

The stabilization of vibrating beam with distributed piezoelectric sensor, actuator, and velocity feedback has been studied. The stabilization of parametric vibrations needs sufficiently large active damping. Admissible variances of loading depend strongly on the feedback gain factor. The saturation effect is observed for large values of gain factor. Double-modes control decreases the almost sure instability regions. Increase of constant components of in-plane forces decreases stability regions. Double-modes control enlarges the almost sure stability regions. Increase of constant tensile component of in-plane force decreases instability regions.

References

Ang, K.K., Reddy, J.N., Wang, C.M., 2000. Displacement control of Timoshenko beams via induced strain actuators. Smart Materials and Structures 9, 981–984.

Chen, L.W., Lin, C.Y., Wang, C.C., 2002. Dynamic stability and control of a composite beam with piezoelectric layers. Composite Structures 56, 97–109.

Dave, D.J., Craig, T.J., 1985. The influence of shear deformation on the natural frequencies of laminated rectangular plates. Composite Structures, 3. Elsevier Applied Science, London and New York, pp. 660–675.

Mannini, A., 1997. Shear deformation effects on thermal buckling of cross-ply composite laminates. Composite Structures 39, 1–10.

Pavlović, R., Kozić, P., Rajković, P., 2001. Influence of transverse shear on stochastic instability of viscoelastic beam. International Journal of Solids and Structures 38, 6829–6837.

Timoshenko, S.P., Gere, J.M., 1961. Theory of Elastic Stability. McGraw-Hill, New York.

Tylikowski, A., 2003. Shear deformation effects on thermally induced instability of laminated plates. Journal of Thermal Stresses 26, 1251–1261.

Wang, C.M., 1995. Timoshenko beam-bending solutions in terms of Euler–Bernoulli solutions. Journal of Engineering Mechanics, Transactions of ASCE 121, 763–765.